p-forms on d-spherical tesselations

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The spectral properties of p–forms on the fundamental domains of regular tesselations of the d–dimensional sphere are discussed. The degeneracies for all ranks, p, are organised into a double Poincaré series which is explicitly determined. In the particular case of coexact forms of rank $(d-1)/2$, for odd d, it is shown that the heat–kernel expansion terminates with the constant term, which equals $(-1)^{p+1}/2$ and that the boundary terms also vanish, all as expected. As an example of the double domain construction, it is shown that the degeneracies on the sphere are given by adding the absolute and relative degeneracies on the hemisphere, again as anticipated. The eta invariant on S^3/Γ is computed to be irrational.
The spectral counting function is calculated and the accumulated degeneracy given exactly. A generalised Weyl-Polya conjecture for p-forms is suggested and verified.

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1. Introduction.

In a recent work, [1], I have looked at p–forms on tesselations of the three–sphere. In this follow–up, I expand on the higher–dimensional aspects of the formalism initiated there. The generating functions are presented for any p–form although I concentrate, for actual ζ–function computations, on (coexact) forms of the middle rank, $p = (d-1)/2$, on the odd d–dimensional factored sphere, S^d/Γ. The reason for this is that the eigenvalues are perfect squares and the expressions for the spectral objects can be taken a long way in terms of known quantities.

The deck group, Γ, is the complete symmetry group of a regular polytope in $n, = d+1$, dimensions. In another terminology, it is a real reflection group. These have all been classified.

This paper should be looked upon as a direct continuation of [1] and, to avoid repetition, I will use, without derivation, any necessary equations and results of this reference As there, the analysis is presented as an example of spectral theory in bounded domains that is easily, and explicitly, managed *via* images. It largely consists of bolting together already existing pieces of knowledge and taking the expressions a little further than seems to exist in the literature.

The quantum field theory of anti–symmetric fields has a certain importance, *e.g.* [2–5]. I will consider a selection of spectral objects, such as heat–kernel expansion coefficients, the Casimir energy and the eta invariant, as examples.

2. Spectrum and generating functions

The coexact eigenvalues of the Hodge de–Rham Laplacian, $d\delta + \delta d$, on the d–sphere are standard,

$$\lambda^{CE}(p,l) = \big(l + (d+1)/2\big)^2 - \big((d-1)/2 - p\big)^2\,, \quad l = 0,1,\dots\,, \tag{1}$$

which specialises to

$$\mu^{CE}(p,l) = \big(l + p + 1\big)^2\,, \quad l = 0,1,\dots\,, \tag{2}$$

for middle rank forms, if d is odd.

The corresponding degeneracies, $g_b^{CE}(p,l)$, are best encoded in the generating function defined by,

$$g_b^{CE}(p,\sigma) \equiv \sum_{l=0}^{\infty} g_b^{CE}(p,l)\,\sigma^l\,, \tag{3}$$

where the label, $b = r$ or a, indicates the conditions satisfied by the p–form on the boundary, $\partial\mathcal{M}$, of the fundamental domain, $\mathcal{M}$, for the action of Γ on S^d. Absolute ('a') conditions arise when the form is symmetric under this action while relative ('r') ones originate from anti–symmetric behaviour. The relation is the duality one,

$$g_b^{CE}\big(p,\sigma\big) = g_{*b}^{CE}\big(d-1-p,\sigma\big)\,, \tag{4}$$

which is a consequence of the $\mathbb{R}^n$ duality,

$$h_{*b}^{CCC}\big(n-p,\sigma\big) = h_b^{CCC}\big(p,\sigma\big)\,, \tag{5}$$

for the closed–coclosed functions, h^{CCC}, and the relation between coexact and closed,

$$g_b^{CE}\big(p,\sigma\big) = h_b^{CCC}\big(p+1,\sigma\big)\,. \tag{6}$$

An important fact is that forms of the middle rank are self–dual in the sense that $g_b^{CE}\big((d-1)/2,\sigma\big) = g_{*b}^{CE}\big((d-1)/2,\sigma\big)$.

Equations are developed in [1] that allow the generating function to be found in closed form in terms of the degrees, $\mathbf{d} = (d_1, d_2, \ldots, d_d)$, that define the polytope (reflection) group, Γ. The case of $p = 1$, $d = 3$ was treated in detail there. Now I expose the general result for *any* coexact p–form in any dimension d,

$$g_a^{CE}\big(p,\sigma\big) = (-1)^{d+1+p}\,\frac{\sum_{q=0}^{d-1-p}(-1)^q e_{d-q}\big(\sigma^{d_1},\ldots,\sigma^{d_d}\big)}{\sigma^{p+1}\prod_{i=1}^d\big(1-\sigma^{d_i}\big)}\,, \tag{7}$$

where the e_q are the elementary symmetric functions.

It is useful to write out the relative generating function from the duality relation (4),

$$\begin{aligned} g_r^{CE}\big(p,\sigma\big) &= (-1)^p\,\frac{\sum_{q=0}^{p}(-1)^q e_{d-q}\big(\sigma^{d_1},\ldots,\sigma^{d_d}\big)}{\sigma^{d-p}\prod_{i=1}^d\big(1-\sigma^{d_i}\big)}\,,\\ &= (-1)^{p+d}\,\frac{\sum_{q=d-p}^{d}(-1)^q e_q\big(\sigma^{d_1},\ldots,\sigma^{d_d}\big)}{\sigma^{d-p}\prod_{i=1}^d\big(1-\sigma^{d_i}\big)}\,. \end{aligned} \tag{8}$$

I derive these expressions later, while developing the formalism.

As in [1], the behaviour under the inversion $\sigma \to 1/\sigma$ is important. From (7),

$$g_a^{CE}\big(p,1/\sigma\big) = (-1)^{p+1}\sigma^{p+1}\,\frac{\sum_{q=0}^{d-1-p}(-1)^q e_q\big(\sigma^{d_1},\ldots,\sigma^{d_d}\big)}{\prod_{i=1}^d\big(1-\sigma^{d_i}\big)}\,, \tag{9}$$

and combined with (8), this gives,

$$T_a^{CE}(p,1/\sigma)-(-1)^d T_r^{CE}(p,\sigma)=\sigma^{p-(d-1)/2}\,(-1)^{p+1}\,, \tag{10}$$

after defining,

$$T_b^{CE}(p,\sigma)=\sigma^{(d+1)/2}\,g_b^{CE}(p,\sigma)\,. \tag{11}$$

For self–dual forms, d is odd and (10) gives the *symmetrical* part of the 'cylinder kernel',

$$T_b^{CE}(p,1/\sigma)+T_b^{CE}(p,\sigma)=(-1)^{p+1}\,,\quad (b=a,\,r)\,, \tag{12}$$

which can be employed to advantage when evaluating the ζ–function in the next section.

3. Zeta functions and heat–kernels

A useful spectral organising quantity is the coexact ζ–function,

$$\zeta_b^{CE}(p,s)=\sum_{l=0}^{\infty}\frac{g_b^{CE}(p,l)}{\lambda^{CE}(p,l)^s}\,,$$

because on a manifold, with or without a boundary, there is the general decomposition,

$$\zeta_b(p,s)=\zeta_b^{CE}(p,s)+\zeta_b^{CE}(p-1,s)\,, \tag{13}$$

of the total ζ–function for the Hodge–de Rham Laplacian.

Only for middle rank forms can ζ^{CE} be related, using (2), to the generating functions and, for the remainder of this section, I will make this simplifying restriction so that $d=2p+1$. The ζ–function is then,

$$\begin{aligned}\zeta_a^{CE}(p,s)&=\frac{i\Gamma(1-2s)}{2\pi}\int_{C_0}d\tau\,(-\tau)^{2s-1}\,T_a^{CE}(p,\tau)\\&=\frac{i\Gamma(1-2s)}{2\pi}\int_{C_0}d\tau\,(-\tau)^{2s-1}\,e^{-\tau(d+1)/2}\,g_a^{CE}(p,\tau)\end{aligned} \tag{14}$$

where I have introduced $\tau=-\log\sigma$ and understand, notationally, $T_a^{CE}(p,\tau)\equiv T_a^{CE}(p,\sigma)$ and $g_a^{CE}(p,\tau)\equiv g_a^{CE}(p,\sigma)$. (I have written the absolute quantity, but this equals the relative one.)

Therefore, from (7),

$$\zeta_a^{CE}(p,s)=(-1)^p\frac{i\Gamma(1-2s)}{2\pi}\int_{C_0}d\tau\,(-\tau)^{2s-1}\sum_{q=0}^{d-1-p}(-1)^q\frac{e_{d-q}\big(e^{-d_1\tau},\ldots,e^{-d_d\tau}\big)}{\prod_{i=1}^d\big(1-e^{-d_i\tau}\big)}\,. \tag{15}$$

I now recall the integral representation of the Barnes ζ–function,

$$\zeta_d(s,a|\mathbf{d})=\frac{i\Gamma(1-s)}{2\pi}\int_{C_0}d\tau\frac{e^{-a\tau}(-\tau)^{s-1}}{\prod_{i=1}^d\big(1-e^{-d_i\tau}\big)}\,, \tag{16}$$

so that (15) becomes,

$$\begin{aligned}\zeta_a^{CE}(p,s)&=(-1)^p\sum_{q=0}^{d-1-p}(-1)^q\sum_{\substack{i_1<i_2<\ldots<i_q\\=1}}^{d}\zeta_d\big(2s,\Sigma d-d_{i_1}-\ldots-d_{i_q}\,|\,\mathbf{d}\big)\\&=(-1)^p\sum_{q=0}^{d-1-p}(-1)^q\sum_{\substack{i_1<i_2<\ldots<i_{d-q}\\=1}}^{d}\zeta_d\big(2s,d_{i_1}+\ldots+d_{i_{d-q}}\,|\,\mathbf{d}\big)\\&=(-1)^{p+d}\sum_{q=d-p}^{d}(-1)^q\sum_{\substack{i_1<i_2<\ldots<i_q\\=1}}^{d}\zeta_d\big(2s,d_{i_1}+\ldots+d_{i_q}\,|\,\mathbf{d}\big)\,,\end{aligned} \tag{17}$$

by a simple reordering.

As an example, I calculate the values $\zeta_a^{CE}(p,-k/2)$, $k\in\mathbb{Z}$. Averaging the first and third lines of (17),

$$\begin{aligned}\zeta_a^{CE}(p,-k/2)=(-1)^p\frac{k!}{2(d+k)!\prod d_i}&\bigg(\sum_{q=0}^{d-1-p}(-1)^{q+k}+\sum_{q=d-p}^{d}(-1)^q\bigg)\\&\times\sum_{\substack{i_1<i_2<\ldots<i_q\\=1}}^{d}B_{d+k}^{(d)}(d_{i_1}+\ldots+d_{i_q}\,|\,\mathbf{d})\,.\end{aligned}$$

If k is even, the two sums combine,

$$\zeta_a^{CE}(p,-k)=\frac{(-1)^p(2k)!}{2(d+2k)!\prod d_i}\sum_{q=0}^{d}(-1)^q\sum_{\substack{i_1<i_2<\ldots<i_q\\=1}}^{d}B_{d+2k}^{(d)}(d_{i_1}+\ldots+d_{i_q}\,|\,\mathbf{d})\,. \tag{18}$$

Special interest is attached to the value $k=0$,

$$\zeta_a^{CE}(p,0)=\frac{(-1)^p}{2d!\prod d_i}\sum_{q=0}^{d}(-1)^q\sum_{\substack{i_1<i_2<\ldots<i_q\\=1}}^{d}B_d^{(d)}(d_{i_1}+\ldots+d_{i_q}\,|\,\mathbf{d})\,. \tag{19}$$

To evaluate these, I note that the Barnes ζ–function satisfies the recursion, [6],

$$\zeta_d(s, a+d_i \,|\, \mathbf{d}) = \zeta_d(s, a \,|\, \mathbf{d}) - \zeta_{d-1}(s, a \,|\, \hat{\mathbf{d}}_i)\,,$$

whose limiting iteration is, [7],

$$\begin{aligned}
\zeta_d(s, a+d_1+\ldots+d_d \,|\, \mathbf{d}) - \sum_{*=1}^{d} &\zeta_d(s, a+d_1+\ldots+*+\ldots+d_d \,|\, \mathbf{d}) \\
&+ \sum_{*=1}^{d}\sum_{*=1}^{d} \zeta_d(s, a+d_1+\ldots+*+\ldots+*+\ldots+d_d \,|\, \mathbf{d}) \\
&\vdots \\
&+ (-1)^{d-1}\sum_{i=1}^{d} \zeta_d(s, a+d_i \,|\, \mathbf{d}) + (-1)^d \zeta_d(s, a \,|\, \mathbf{d}) \\
&= a^{-s}\,.
\end{aligned} \tag{20}$$

In the first summation, the star denotes that one of the d's is to be omitted, in turn. In the second summation, every two different pairs of d's must be successively omitted, and so on. This is the notation of Barnes [6]. The star summations and omissions correspond to the more conventional ordered summations in *e.g.* (17).

The iteration, (20), on setting s to specific values or by extracting the poles of the ζ–function, leads to identities involving the generalised Bernoulli polynomials. For example, setting s equal to $-2k$, it is possible to safely equate a to zero and (20) gives the values of the expressions (18) and (19). I find,

$$\begin{aligned}
\zeta_a^{CE}(p, -k) &= 0 \\
\zeta_a^{CE}(p, 0) &= \frac{1}{2}\,(-1)^{p+1}\,,
\end{aligned} \tag{21}$$

which show that the coexact middle form heat–kernel coefficients, $C^{CE}_{k+d/2}$, $k = 1, 2, \ldots$, vanish and that the constant term, $C^{CE}_{d/2}$, equals $(-1)^{p+1}/2$. This derivation is related to, but independent of, that presented in [1]. It does not depend on the fact that the $C_{d/2}(p)$ coefficient for a general p–form vanishes on the *doubled* fundamental domain,

$$C^b_{d/2}(p) + C^{*b}_{d/2}(p) = 0\,, \quad \forall p\,. \tag{22}$$

and could be taken as a proof of this fact.

Furthermore, as mentioned at the end of the previous section, the symmetrical part of the integrand, (12), produces (21) immediately, bypassing explicit use of the recursion formulae which I have given, though, for completeness.

The heat–kernel expansion terminates with the constant term, which generalises the result in [8] on the full sphere. It is almost obvious that factoring the sphere will not alter this fact. The only question would be the effect of the fixed points.

Adding the two (equal) constants for absolute and relative conditions gives the constant for the doubled fundamental domain. In particular, the value, $(-1)^{p+1}$, holds for the full sphere. This agrees with the known value, [8–10].

The remaining heat–kernel coefficients, $C^{CE}_{k/2}$, follow from the 'positive' poles of the coexact ζ–function at $s=(d-k)/2$, for $k=0,1,\ldots,d-1$, which themselves result from the known poles of the Barnes ζ–function according to (17),

$$C^{CE}_{k/2}=(-1)^p\frac{\Gamma\big((d-k)/2\big)}{2k!(d-k-1)!\prod d_i}\bigg(\sum_{q=0}^{d-1-p}(-1)^q-\sum_{q=d-p}^{d}(-1)^{q+k}\bigg)$$
$$\times\sum_{\substack{i_1<i_2<\ldots<i_q\\=1}}^{d}B^{(d)}_k(d_{i_1}+\ldots+d_{i_q}\,|\,\mathbf{d})\,.$$

If k is odd, the summations combine to allow the pole part of (20) to come into use showing that the boundary coefficients, *i.e.* $C^{CE}_{k/2}$ for k odd, are zero, again generalising the result in [1]. As before, this conclusion follows more easily from (12).

There are no such 'topological' simplifications or cancellations for the other values of the ζ–function, for example for the coexact Casimir energy,

$$\begin{aligned}E&=\frac{1}{2}\,\zeta^{CE}_a(p,-1/2)\\&=\frac{(-1)^p}{2(d+1)!\prod d_i}\sum_{q=0}^{p}(-1)^q\sum_{\substack{i_1<i_2<\ldots<i_q\\=1}}^{d}B^{(d)}_{d+1}\big(d_{i_1}+\ldots+d_{i_q}\,|\,\mathbf{d}\big)\,,\end{aligned}$$

and one is reduced to actual computation.

4. Extensions and elaborations

Although the expression, (14), for $\zeta^{CE}_b(p,s)$ is valid just for the middle rank forms on S^d/Γ, it has significance for *all* p when the factored sphere is realised as the base of a generalised cone in $\mathbb{R}^{d+1}$, [11], [12]. I have also referred to this construction as a bounded Möbius corner, [1,13]. The separation of variables into

radial and angular introduces a term that effectively cancels the second part of (1). Gilkey, [14] §4.7.5, refers to the resulting operator as the *normalised* spherical Laplacian.

In this case the T quantities defined in (11) are *bone fide* cylinder kernels (without propagation significance) and the corresponding absolute ζ–function is,

$$\begin{aligned}
\zeta_a^{CE}(p,s) &= \frac{i\Gamma(1-2s)}{2\pi}\int_{C_0} d\tau\,(-\tau)^{2s-1}\,T_a^{CE}(p,\tau)\\
&= (-1)^p\frac{i\Gamma(1-2s)}{2\pi}\int_{C_0} d\tau\,(-\tau)^{2s-1}e^{-\big((d-1)/2-p\big)\,\tau}\;\times\\
&\qquad\qquad\sum_{q=0}^{d-1-p}(-1)^q\frac{e_{d-q}\big(e^{-d_1\tau},\ldots,e^{-d_d\tau}\big)}{\prod_{i=1}^d\big(1-e^{-d_i\tau}\big)}\\
&= (-1)^d\sum_{q=p+1}^{d}(-1)^q\sum_{\substack{i_1<i_2<\ldots<i_q\\=1}}^{d}\zeta_d\big(2s,\frac{d-1}{2}-p+d_{i_1}+\ldots+d_{i_q}\,|\,\mathbf{d}\big)\,,
\end{aligned}\tag{23}$$

with duality giving the relative $\zeta_r(p,s)=\zeta_a(d-1-p,s)$,

$$\zeta_r^{CE}(p,s)=(-1)^d\sum_{q=d-p}^{d}(-1)^q\sum_{\substack{i_1<i_2<\ldots<i_q\\=1}}^{d}\zeta_d\big(2s,p-\frac{d-1}{2}+d_{i_1}+\ldots+d_{i_q}\,|\,\mathbf{d}\big)\,.\tag{24}$$

These ζ–functions appear as useful intermediate quantities but have no independent dynamical significance. In our work on the ball, Dowker and Kirsten [11], they were referred to as 'modified' ζ–functions. The present results would allow us to extend the ball calculations to factored bases in a systematic fashion. For example the computations of the scalar functional determinants reported in [13] could be generalised to p–forms.

An expression for the modified ζ–function on the full sphere is given in equn. (42) in [11]. A related formula can be obtained from our present results by adding the absolute and relative expressions on the hemisphere, for which all the degrees are one. Hence from (23) and (24),

$$\begin{aligned}
\zeta_{sphere}^{CE}(p,s) = (-1)^d\bigg(&\sum_{q=p+1}^{d}(-1)^q\binom{d}{q}\zeta_d\big(2s,\frac{d-1}{2}-p+q\,|\,\mathbf{d}\big)+\\
&\sum_{q=d-p}^{d}(-1)^q\binom{d}{q}\zeta_d\big(2s,p-\frac{d-1}{2}+q\,|\,\mathbf{d}\big)\bigg)\,,
\end{aligned}$$

which, after some manipulation, is equivalent to the form given in [11].

More generally, adding relative and absolute produces the results for a 'doubled' fundamental domain. Rather than give the full expressions, I will only look at the consequences of the duality relation, (10) which yields,

$$T^{CE}_{a+r}(p,\tau)-(-1)^dT^{CE}_{a+r}(p,-\tau)=2(-1)^{p+d}\cosh\big(p-(d-1)/2\big)\tau\,,$$

allowing ζ–function values to be easily found as powers of $p-(d-1)/2$. I will not do this in detail and only remark that these values are independent of the factoring, Γ.

5. The eta invariant

An important spectral quantity is the eta invariant, $\eta(0)$, which gives a measure of the asymmetry of the spectrum. Originally introduced by Atiyah, Patodi and Singer, [15], as a boundary 'correction' to an index, it has achieved an independent life, and its computation has become a standard challenge. A number of approaches, simple and sophisticated, are available. The original one, [15], employs the G–index theorem. According to Donnelly, [16], Millson was the first to evaluate $\eta(0)$ on lens spaces by direct calculation. A direct spectral computation, in the particular case of spherical space forms, is also mentioned in [15] and attributed to Ray. Such a calculation was given, later, by Katase, [17], on quotients of the 3–sphere. The analysis is somewhat involved and the result is just the general angle form given previously. Even so, I outline my own version below.

The actual numbers for the various homogeneous (fixed point free) quotients of S^3 were computed by Gibbons *et al*, [18], who performed the group average by summing over the angles that define the elements. Something similar was done by Seade [19]. In [20], I offered an algebraic alternative to this rather cumbersome geometric technique. The eta invariant on spherical space forms has been systematically investigated by Gilkey [14,21].

The eta function, $\eta(s)$, measures the asymmetry of the boundary part of an operator and, as such, is computed on a closed manifold. As an exercise, I wish to find it for fundamental domains associated with the quotient S^d/Γ and these have a boundary. In fact, things are not quite so bad because I can work on the doubled fundamental domains resulting from the restriction to the direct rotational polytope group. However, the domain does have edges and vertices.

The fundamental domain, $\mathcal{M}$, of the spherical tesselation is the base of the generalised (metric) cone formed, on $\mathbb{R}^n$, by the set of reflecting hyperplanes that define the extended group Γ. As such, it is part of the boundary of this cone, or Möbius corner (kaleidoscope), the other part being the union of the flat sides. Restricting to the rotational subgroup of Γ turns the cone into a periodic one whose boundary is just its base, the doubled fundamental domain, $2\mathcal{M}$. There are, however, singularities of codimension two, corresponding to the edges of the fundamental domain, and of codimension three from the vertices, *cf* [22].

The signature eta function on a d–manifold, N, (typically a boundary) is neatly expressed in terms of the middle rank coexact eigenforms, ϕ_l, by, [23],

$$\eta(s) = \sum_l \int_N \frac{\phi_l^* \wedge d\phi_l}{\mu_l^{s+1/2}} \,,$$

where $\mu_l = \mu^{CE}(p,l)$ of (2), and $p = (d-1)/2$ is odd, $(d = 4D-1)$. For each label, l, there are, possibly, two coexact eigenforms ('positive' and 'negative') that can be chosen[2] to be eigenforms of $*d$,

$$*d\,\phi_l = \pm\,\omega_l\,\phi_l\,, \quad \omega_l = \sqrt{\mu}_l\,,$$

and the sum is over both types. The ω spectrum is not generally symmetric.

Despite my preference for the algebraic method, since all the hard work has been done on the 3–sphere, I initially use angle summation. Only the signature eta function will be considered and I now derive, again, the expression obtained long ago in [15].

In physicist's language, the signature eta function in four dimensions is just the transverse spin–one spectral asymmetry function on the three–dimensional boundary, [24,25]. For spherical factors, the necessary spectral information has been given a number of times before in various connections, *e.g.* [26,27], and repeated in [20,28].

In terms of the positive and negative, spin–1 'Hamiltonian' ζ–functions, on S^3/Γ,

$$\begin{aligned}\zeta^+(s) &= \sum_{\overline{L}=1}^\infty \frac{d^+(\overline{L})}{(\overline{L}+1)^s}\\ \zeta^-(s) &= \sum_{\overline{L}=3}^\infty \frac{d^-(\overline{L})}{(\overline{L}-1)^s} = \sum_{\overline{L}=1}^\infty \frac{d^-(\overline{L}+2)}{(\overline{L}+1)^s}\,,\end{aligned} \tag{25}$$

[2] The restriction $d = 4D-1$ is necessary for this. I have also allowed for complex eigenforms, although they can be arranged to be real. I have not distinguished notationally between the positive and negative types.

the eta function is defined to be,

$$\eta(s) = \zeta^+(s) - \zeta^-(s)\,.$$

The degeneracies, $d^\pm$, follow from character theory and are given in the just cited references. The eta function can be written as the group average,

$$\eta(s) = \frac{1}{|\Gamma|}\sum_\gamma \eta(\gamma, s)\,, \tag{26}$$

where, by the algebra detailed in [28], the partial eta function, $\eta(\gamma, s)$, is,

$$\eta(\gamma, s) = 2\sum_{n=1}^\infty \frac{1}{n^s}\,\frac{\sin\alpha\sin n\beta - \sin\beta\sin n\alpha}{\cos\alpha - \cos\beta}\,. \tag{27}$$

The sum over γ in (26) is a sum over the angles, α and β.

The important value is $\eta(0)$ and substitution into (27) shows that there are no problems with the fixed points. As usual, the identity element gives zero as do the other special values, $\alpha = 0$, $\beta \neq 0$, which correspond to the fixing of a 2–flat in the ambient $\mathbb{R}^4$. The summation over n is then trivially performed using $2\sum_{n=1}^\infty \sin n\theta = \cot\theta/2$ giving,

$$\eta(0) = -\frac{1}{|\Gamma|}\sum_{\alpha\neq 0;\,\beta} \cot\alpha/2\,\cot\beta/2\,, \tag{28}$$

which is, apart from the summation restriction, the standard formula [3].

The values of α and β corresponding to the elements of the several polytope groups can now be inserted and the group average performed using the class decompositions given in [1] which were taken from Hurley [29] and Chang [30]. I find the values for the doubled fundamental domain,

$$\begin{aligned}
\eta(0) &= -\frac{2}{5\sqrt5}\,, && \{3^3\}\\
&= -\frac{5}{16}\,, && \{3^2 4\}\\
&= -\frac{29}{48}\,, && \{343\}\\
&= -\frac{2341}{5400} - \frac{118}{75\sqrt5}\,, && \{3^2 5\}\,.
\end{aligned}$$

[3] This derivation is, no doubt, the same as those of Millson and Ray mentioned earlier.

The novelty is the presence of the surd in two cases and the simple fractions in the others. By contrast, in the evaluation of the Casimir energy all irrationalities cancel.

The eta function on a lune is, almost trivially, zero because all group elements fix a 2–flat 'axis' of rotation, and these contribute nothing.

Without going through the mode analysis, it is reasonable that the Dirac eta invariant will be given by the standard, basic expression as given, in Hanson and Römer, [31], *e.g.*, see [32],

$$\eta_S(0) = -\frac{1}{2|\Gamma|}\sum_{\alpha\neq 0;\,\beta} \operatorname{cosec}\alpha/2\,\operatorname{cosec}\beta/2\,. \tag{29}$$

Numerical evaluation yields the following values,

$$\begin{aligned}
\eta_S(0) &= -\frac{1}{5\sqrt{5}}\,, && \{3^3\}\\
&= -\frac{89}{768}-\frac{9}{32\sqrt{2}}\,, && \{3^2 4\}\\
&= -\frac{1867}{1728}-\frac{9}{8\sqrt{2}}\,, && \{343\}\\
&= -\frac{37291}{7200}+\frac{277}{75}\frac{1}{\sqrt{5}}\,, && \{3^2 5\}\,.
\end{aligned}$$

The presence of the surds implies that it is not possible to find alternative expressions for the eta invariant purely in terms of the degrees, d_i, as it is for *homogeneous*, fixed point free quotients, [20], or for the Casimir energy.

Incidentally, this conclusion seems not in agreement with the work of Degeratu, [33], which relates the coefficients in the Laurent expansion of the Molien (Poincaré) series directly to the (Dirac) eta invariants associated with the boundaries, S^3/Γ_i, of the orbifolds, $\mathbb{C}^2/\Gamma_i$ where the Γ_i are subgroups of Γ.

Cheeger, [23], discusses the eta invariant on a generalised cone. For the standard situation of a smooth manifold, M, a generalised cone is attached to the boundary, N, converting M into X, a compact space, with a conical singularity, on which the index can be calculated. In this way, Cheeger shows that the standard Atiyah–Patodi–Singer formula for $\mathrm{Sig}(M)$ follows from spectral analysis on the cone, the boundary $\eta(0)$ arising now from the effect of the cone apex. Also mentioned is the non–standard eta function on manifolds with boundaries or with conical points and the possibility that it might be irrational is raised. My computation seems to confirm this and I leave it at this point.

6. Developing the formalism – the double Poincaré series

In this section I present a derivation of my basic formulae, (7) and (8), from the recursions for the various generating functions given in [1] which are defined as sums over the mode label, as in (3). Although not necessary, I will do this using double generating functions obtained from the previous ones by summing also over the form rank, p. Such double series are used by Ray on spheres, [34], but my approach is different in detail and, in fact, refers to the expressions *after* the group average. They allow for a compressed treatment.

I start with the degeneracy of harmonic polynomial forms on $\mathbb{R}^n$ and define the double Poincaré series, a finite, 'fermionic' polynomial in z,

$$h_b(z,\sigma) = \sum_{p=0}^{n} h(p,\sigma)\, z^p = \frac{1-\sigma^2}{|\Gamma|} \sum_A \frac{\det(1+zA)}{\det(1-\sigma A)}\, \chi^*(A)\,. \tag{30}$$

It is possible to think of z as a fugacity.

Using standard identities, explicit forms are,

$$h_a(z,\sigma) = (1+z\sigma) \prod_{i=1}^{d} \frac{1+z\sigma^{m_i}}{1-\sigma^{d_i}}\,, \tag{31}$$

and

$$\begin{aligned} h_r(z,\sigma) &= (z+\sigma) \prod_{i=1}^{d} \frac{z+\sigma^{m_i}}{1-\sigma^{d_i}} \\ &= z^n\, h_a\big(\frac{1}{z},\sigma\big)\,, \end{aligned} \tag{32}$$

where the final factor in the numerator has been extracted using $m_n = 1$. Equation (30), without the harmonic factor $(1-\sigma^2)$, is Solomon's theorem, [35], Bourbaki, [36] p.136, Kane, [37] §22.4,. See also, relatedly, Sturmfels, [38], p.37, Exercise 5. The explicit forms are algebraic while the group average is geometric.

I introduce the notion of polynomial dual, or reciprocal, by the definition,

$${}^*f(z) \equiv z^n\, f(1/z)\,, \quad i.e.\ \ {}^{**}f(z) = f(z)\,,$$

on a polynomial, f, of (unwritten) degree n, so that, *e.g.*, the relation (32) reads,

$$h_{*b}(z,\sigma) = {}^*h_b(z,\sigma)\,. \tag{33}$$

This helps notationally when dealing with the maximum form rank, $p = n$, on $\mathbb{R}^n$.

The recursions given in [1] transcribe into formulae that can be solved algebraically. I give the basic steps. For example, the harmonic and closed harmonic generating functions are related by the recursion,

$$h_b(p,\sigma) = h_b^C(p,\sigma) + \sigma h_b^C(p+1,\sigma) \tag{34}$$

which, on account of $h_b^C(p,\sigma) = 0$ $(p > n)$, becomes, using the same basic symbols,

$$h_b(z,\sigma) = \overline{h}_b^C(z,\sigma) + \frac{\sigma}{z}\left(\overline{h}_b^C(z,\sigma) - \overline{h}_b^C(0,\sigma)\right), \tag{35}$$

where $\overline{h}$ is defined by,

$$\overline{h}_b^C(z,\sigma) = h_b^C(z,\sigma) - \sigma^2\delta_{ba}\,. \tag{36}$$

In going from (34) to (35), the term involving σ^2 has been inserted by hand, *via* (36), for the reason mentioned in [1] for adding a term, $\delta_{ba}\delta_{p0}\,\delta_{l2}$, to the solution of the recursion for $h_b^C(p,l)$. It is needed to ensure the required zero mode end point value, $h_b^C(0,\sigma) = \delta_{ba}$, which is not covered by the exact sequence that provides the recursion.

It is useful at this point to list some of the end point values,

$$\begin{gathered}
h_b^C(0,\sigma) = \delta_{ba}, \quad {}^*h_b^C(0,\sigma) = {}^*h_b(0,\sigma), \quad h_b^{CCC}(0,\sigma) = \delta_{ba}\,, \\
{}^*h_b^{CCC}(0,\sigma) = \delta_{br}, \quad {}^*h_b^{CC}(0,\sigma) = \delta_{br}, \quad h_b^{CC}(0,\sigma) = h_b(0,\sigma)\,.
\end{gathered} \tag{37}$$

The identities established in [1] also take on an elegant appearance. For example the supertraces,

$$h_b(-\sigma,\sigma) = \delta_{ba}(1-\sigma^2)\,, \quad h_b^{CC}(-\sigma,\sigma) = \delta_{ba}\,. \tag{38}$$

The first equation is actually an easy consequence of (30).

Thus, setting $z = -\sigma$ in (35), gives,

$$h_b(-\sigma,\sigma) = \overline{h}_b^C(0,\sigma) = (1-\sigma^2)\,\delta_{ba}\,,$$

as an algebraic check.

The solution of (35) is,

$$h_b^C(z,\sigma) = \frac{z}{z+\sigma}\,h_b(z,\sigma) + \frac{1+z\sigma}{1+z/\sigma}\,\delta_{ba}\,. \tag{39}$$

For the closed and the closed–coclosed functions there are no 'correction terms' and the recursion formula, which has the same form as (35), gives,

$$h_b^{CCC}(z,\sigma) = \frac{z}{z+\sigma}\;h_b^{CC}(z,\sigma) + \frac{\sigma}{z+\sigma}\,\delta_{ba}\,, \tag{40}$$

using $h_b^{CCC}(0,\sigma)=\delta_{ba}$.

The left–hand side of (40) is the quantity required as it encodes the closed degeneracies on the d–sphere, [39]. The inputs are the explicit forms (31) and (32) which can be used after relating h^{CC} and h^C by duality on $\mathbb{R}^n$. This yields the various equivalent forms,

$$\begin{aligned}
h_b^{CC}(z,\sigma) &= z^n\, h_{*b}^C\big(\frac{1}{z},\sigma\big) \,=\, {}^*h_{*b}^C(z,\sigma)\\
&= \frac{z^n}{1+z\sigma}\, h_{*b}\big(\frac{1}{z},\sigma\big)+\frac{z^n\sigma(z+\sigma)}{1+z\sigma}\,\delta_{*ba}\\
&= \frac{1}{1+z\sigma}\, h_b\big(z,\sigma\big)+\frac{z^n\sigma(z+\sigma)}{1+z\sigma}\,\delta_{br}\,.
\end{aligned}\tag{41}$$

and

$$ {}^*h_b^{CC}(z,\sigma)=h_{*b}^C(z,\sigma)=\frac{z}{z+\sigma}\,h_{*b}\big(z,\sigma\big)+\frac{1+z\sigma}{1+z/\sigma}\,\delta_{br}\,, \tag{42}$$

using (39). These expressions exhibit immediately the end values in (37).

Substitution into (40) produces,

$$h_b^{CCC}(z,\sigma)=\frac{1}{(1+\sigma/z)(1+\sigma z)}\,h_b(z,\sigma)+\frac{z^n z\sigma}{1+z\sigma}\,\delta_{br}+\frac{\sigma}{z+\sigma}\,\delta_{ba}\,. \tag{43}$$

For consistency, duality on $\mathbb{R}^n$ in the form,

$$h_b^{CCC}(z,\sigma)=\,{}^*h_{*b}^{CCC}(z,\sigma)\,, \tag{44}$$

can easily be checked. Hence it is sufficient to restrict to absolute conditions and the explicit form (31) results in the final expression,

$$h_a^{CCC}(z,\sigma)=\frac{z}{z+\sigma}\prod_{i=1}^d\frac{1+z\sigma^{m_i}}{1-\sigma^{d_i}}+\frac{\sigma}{z+\sigma}\,, \tag{45}$$

which is equivalent to (7), taking into account the relation (6).

One way of showing this, is to go backwards and derive the recursion satisfied by the double Poincaré series constructed directly from (7), virtually paralleling the previous analysis. For simplicity, I rename $h_a^{CCC}(*,\sigma)$ as $H_a(*)$.

From (7) it is straightforward to derive the form rank recursion,

$$\sigma^{p+1}H_a(p+1)+\sigma^p\,H_a(p)=\frac{e_p(\sigma^{d_1},\ldots,\sigma^{d_d})}{\prod_{i=1}^d(1-\sigma^{d_i})}\,. \tag{46}$$

I next note that, because of the end point value ${}^*H(0)=0$, the top limit in the sum defining $H(z)$ can be put at $p=d$ (appropriate for working on S^d) and the construction of the generating function of z allows the recursion (46) to be rewritten and then solved, much as before, to give,

$$H_a(\sigma z)=\frac{z}{1+z}\prod_{i=1}^{d}\frac{1+z\sigma^{d_i}}{1-\sigma^{d_i}}+\frac{1}{1+z}\,,$$

which is the same as (45) after setting $z\to z/\sigma$, moreover, one can see how the exponents, m_i, turn into the degrees, d_i. In this, slightly synthetic way, I have justified the forms (7) and (8). A direct demonstration is possible.

For convenience I give the expression for the absolute coexact double generating function that follows from (45) and the relation (6),

$$g_a^{CE}(z,\sigma)=\frac{1}{z+\sigma}\bigg[\prod_{i=1}^{d}\frac{1+z\sigma^{m_i}}{1-\sigma^{d_i}}-1\bigg]\,, \tag{47}$$

a neat encapsulation of my results. For completeness, the relative version reads [4],

$$\begin{aligned}g_r^{CE}(z,\sigma)&=\frac{1}{1+z\sigma}\bigg[\prod_{i=1}^{d}\frac{z+\sigma^{m_i}}{1-\sigma^{d_i}}+z^n\sigma\bigg]\\&=z^{d-1}\bigg(g_a^{CE}(\frac{1}{z},\sigma)+z\bigg)\,.\end{aligned} \tag{48}$$

I remark again that the g^{CE} are polynomials in z.

On the d–hemisphere,

$$g_a^{CE}(z,\sigma)\bigg|_{hemisphere}=\frac{1}{z+\sigma}\bigg[\bigg(\frac{1+z}{1-\sigma}\bigg)^d-1\bigg]\,, \tag{49}$$

and

$$z^d g_r^{CE}(\frac{1}{z},\sigma)\bigg|_{hemisphere}=\frac{z}{z+\sigma}\bigg[\bigg(\frac{1+z}{1-\sigma}\bigg)^d+\frac{\sigma}{z}\bigg]\,, \tag{50}$$

while a supertrace result is,

$$\sigma\, g_a^{CE}(-\sigma,\sigma)+d=\sum_{i=1}^{d}\frac{1}{1-\sigma^{d_i}}\,.$$

[4] When checking the various relations it is necessary to note that, algebraically, $g_a^{CE}(p,\sigma)=1$ for $p=-1$.

7. The 0–form case

When $z = 0$, *i.e.* $p = 0$, the above expressions for $g^{CE}(z,\sigma)$ do not give the degeneracies for the *general* 0–form. These are, of course, known, but the most convenient way of including them in the present formalism is to extend the range of l down to -1, which corresponds to a *zero mode*, as is apparent from (1). This mode exists only for absolute (*i.e.* Neumann) conditions and is a uniform function, a polynomial of zero order. Hence the *complete* 0–form generating function is,

$$h_b(\sigma) = \delta_{ba} + \sigma\, g_b^{CE}(0,\sigma)\,,$$

which, from (47) and (48), yield the known forms,

$$\begin{aligned} h_a(\sigma) = h_N(\sigma) &= \frac{1}{\prod_{i=1}^d (1-\sigma^{d_i})} \\ h_r(\sigma) = h_D(\sigma) &= \frac{\sigma^{d_0}}{\prod_{i=1}^d (1-\sigma^{d_i})}\,, \quad d_0 = \Sigma_i\, m_i\,. \end{aligned}$$

8. The double fundamental domain and a check

Formulae (47) and (48) give the generating functions on the fundamental domain, $\mathcal{M}$. That on the doubled domain, $2\mathcal{M}$, is obtained by adding these two expressions. This can be checked explicitly for the hemisphere since the full sphere degeneracies are standard, [39]. To this end, I split these as in [11],

$$g^{CE}(p,l) = \frac{(l+d)!}{p!\,(d-p-1)!\,l!}\left(\frac{1}{l+1+p} + \frac{1}{l+d-p}\right), \quad l = 0,1,\dots\,, \tag{51}$$

the two parts of which are related by $p \to d-1-p$. Each part corresponds to a hemisphere contribution. Assuming, for a moment, that this is true, the combinatorial identity used in [11], equn.(41), allows the eigenvalue generating functions on the hemisphere to be found from (51). Then,

$$\begin{aligned} g_a^{CE}(p,\sigma)\Big|_{hemisphere} &= \sum_{m=p+1}^{d} \binom{m-1}{p} \frac{1}{(1-\sigma)^m} \\ g_r^{CE}(p,\sigma)\Big|_{hemisphere} &= \sum_{m=d-p}^{d} \binom{m-1}{d-p-1} \frac{1}{(1-\sigma)^m}\,. \end{aligned}$$

Construction of the double Poincaré series turns these into (49) and (50) by simple algebra, (replace the lower limits by zero), confirming that the two parts do give the two hemisphere contributions. That is,

$$g^{CE}(z,\sigma)\bigg|_{sphere} = g_a^{CE}(z,\sigma)\bigg|_{hemisphere} + g_r^{CE}(z,\sigma)\bigg|_{hemisphere},$$

which is a special case of,

$$g^{CE}(z,\sigma)\bigg|_{2\mathcal{M}} = g_a^{CE}(z,\sigma)\bigg|_{\mathcal{M}} + g_r^{CE}(z,\sigma)\bigg|_{\mathcal{M}},$$

a p–form generalisation of the known scalar result.

As well as the sum of relative and absolute expressions, the difference is also of interest (see the next section). It is preferable to remove the z^d term in (48) and define the combination,

$$w(z,\sigma) = g_r(z,\sigma) - g_a(z,\sigma) - z^d \tag{52}$$

which is a polynomial of degree $d-1$ and is anti–reciprocal,

$$^*w(z,\sigma) = -w(z,\sigma)\,. \tag{53}$$

9. The counting function

The counting function, $N(\lambda)$, could be considered the basic global spectral object[5]. In general terms, let the eigenvalues, λ_i, be ordered linearly $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_i \le \ldots$ with i a counting label and degeneracies accounted for by equality. Then a definition of $N(\lambda)$ is

$$N(\lambda) = \sum_{\lambda_i<\lambda} 1 + \sum_{\lambda_i=\lambda} \frac{1}{2}\,. \tag{54}$$

Alternatively, if the spectrum is described by the distinct *eigenlevels* $\lambda(n)$, $n = 0,1,2,\ldots$, with explicit degeneracies, $g(n) = g\big(\lambda(n)\big)$, *cf* Baltes and Hilf [40],

$$N(\lambda) = \sum_{\lambda(n)<\lambda} g(n) + \frac{1}{2}\,g(n)\,\delta_{\lambda,\lambda(n)}\,. \tag{55}$$

[5]Probably it first appears in Sturm's 1829 treatment of the roots of the secular equation

In the case under consideration in this paper, n is l, the polynomial order, and has a dynamical significance. The eigenvalues are functions of l, hence the notation.

The value of $N(\lambda)$ depends on the particular function $\lambda(n)$ in the sense that, for a fixed argument, λ, it will vary if the form of $\lambda(n)$ is changed. One could, for example, add a variable constant to the eigenvalues. On the other hand, the evaluation of $N(\lambda)$ at an eigenvalue, $G(n) \equiv N\big(\lambda(n)\big)$, is the accumulated degeneracy and does not depend on the form of $\lambda(n)$,

$$G(n) = \sum_{n'=0}^{n} g(n') + \frac{1}{2} g(n) \,.$$

$G(n)$ satisfies the recursion,

$$G(n) - G(n-1) = \frac{1}{2}\big(g(n) + g(n-1)\big)$$

which translates into the relation,

$$G(\sigma) = \frac{1}{2}\left(\frac{1+\sigma}{1-\sigma}\right) g(\sigma) \,. \tag{56}$$

between generating functions, $G(\sigma) = \sum_{n=0}^{\infty} G(n)\sigma^n$ *etc.*

This equation can be applied to the situation in this paper and, for example, from (47) I find,

$$G_a^{CE}(\sigma) = \frac{1}{2}\left(\frac{1+\sigma}{1-\sigma}\right)\frac{1}{z+\sigma}\left[\prod_{i=1}^{d} \frac{1+z\sigma^{m_i}}{1-\sigma^{d_i}} - 1\right] . \tag{57}$$

It is possible to extract the individual accumulated degeneracies[6] by writing,

$$G(l) = \frac{1}{2\pi i} \oint_C d\sigma \, \frac{1}{\sigma^{l+1}} \, G(\sigma)$$

where the contour C circles the origin. The singularities of $N(\sigma)$ lie only at $\sigma = 1$, and it converges as $|\sigma| \to \infty$, so the contour can be deformed into one, C', around $\sigma = 1$,

$$G(l) = -\frac{1}{2\pi i} \oint_{C'} d\sigma \, \frac{1}{\sigma^{l+1}} \, G(\sigma) \,,$$

[6] The degeneracies themselves can be extracted but will not be exhibited here.

and the calculation is one of residues. In principle this gives an explicit expression for $N(l)$. As an example, I treat the hemisphere using (49) and (56),

$$\begin{aligned}
G_a^{CE}(l) &= -\frac{1}{4\pi i}\oint_{C'} d\sigma\,\frac{1}{\sigma^{l+1}}\,\frac{1+\sigma}{(1-\sigma)(z+\sigma)}\left[\left(\frac{1+z}{1-\sigma}\right)^d - 1\right] \\
&= \frac{(-1)^d}{2d!}\,(1+z)^d\,\frac{d^d}{d\sigma^d}\,\frac{1+\sigma}{(z+\sigma)\sigma^{l+1}}\bigg|_{\sigma=1} - \frac{1}{1+z} \\
&= \sum_{r=0}^{d} A_r^d(z)(l+1)(l+2)\ldots(l+r) - \frac{1}{1+z}\,,
\end{aligned} \tag{58}$$

where,

$$\begin{aligned}
A_r^d(z) &= \frac{(-1)^{d-r}}{2\,d!}\binom{d}{r}(1+z)^d\,\frac{d^{d-r}}{d\sigma^{d-r}}\,\frac{1+\sigma}{z+\sigma}\bigg|_{\sigma=1} \\
&= (-1)^{d-r}\,\frac{(1+z)^d}{2(d-r)!r!}\,\frac{d^{d-r}}{d\sigma^{d-r}}\left(1+\frac{1-z}{z+\sigma}\right)\bigg|_{\sigma=1} \\
&= \frac{1}{d!}(1+z)^{d-1}\,, \quad r=d \\
&= \frac{1}{2r!}(1-z)(1+z)^{r-1}\,, \quad r<d\,.
\end{aligned} \tag{59}$$

Separating the $r=d$ and $r=0$ terms, (58) reads,

$$\begin{aligned}
G_a^{CE}(l) = \frac{(l+1)\ldots(l+d)}{d!}\,(1+z)^{d-1} - \frac{1}{2} &+ \\
\frac{1-z}{2}\sum_{r=1}^{d-1}\frac{(1+z)^{r-1}}{r!}\,&(l+1)(l+2)\ldots(l+r)\,,
\end{aligned} \tag{60}$$

which is exact and is, correctly, a finite polynomial in z.

It should be noted that, because of the final term in the particular definition of the counting function, (54), $G(l)$ is half an integer.

For a given eigenvalue form, such as (1), the counting function, $N(\lambda)$ can be determined by the condition that, if $\lambda(l) \le \lambda < \lambda(l+1)$ then $N(\lambda) = G(l)$.

Particular interest lies in the asymptotic behaviour of $N(\lambda)$ as $\lambda \to \infty$, in relation to Weyl's conjecture. It is tolerably clear that the leading term will follow from (60) as $l \to \infty$ since, from (1) $\lambda \sim l^2$ for very large λ. The highest power of l is

$$G_a^{CE}(l) \sim \frac{l^d}{d!}\,(1+z)^{d-1}$$

which corresponds, after slight manipulation, to Weyl's term

$$N(\lambda) \sim \binom{d-1}{p}\frac{|\mathcal{M}|}{(4\pi)^{d/2}}\,\frac{1}{\Gamma(1+d/2)}\,\lambda^{d/2}$$

for coexact p–forms when $\mathcal{M}$ is the hemisphere. With more algebraic work, these calculations can be extended to the general tessellation (57).

The point here is that this leading term depends only on the degeneracies and not on the specific eigenvalues, so long as $\lambda = l^2 + o(l)$. It is possible to go further and derive a form generalisation of the Weyl–Polya conjecture which says that, in the scalar case, $N_D(\lambda) \le N_N(\lambda)$. Proceeding as for the scalar case in Bérard and Besson, [41], by analogy to the definition, (52), I define, the modified difference of accumulated degeneracies,

$$W(z,\sigma) = \frac{1}{2}\frac{1+\sigma}{1-\sigma}\, w(z,\sigma) = G_r^{CE}(z,\sigma) - G_a^{CE}(z,\sigma) - \frac{1}{2}\frac{1+\sigma}{1-\sigma}\, z^d \, .$$

This is an anti–reciprocal polynomial of degree $d-1$ in z,

$$^*W(z,\sigma) = -W(z,\sigma)\,,$$

a statement of duality.

Hence, writing the polynomial as,

$$W(z,\sigma) = w_0 + w_1 z + \ldots + w_{d-1} z^{d-1}\,,$$

one has, $w_i = -w_{d-1-i}$ and, if d is odd, the middle coefficient $w_{(d-1)/2}$ is zero corresponding to the self–duality of the middle rank form discussed earlier.

It is then easy to show that $W(z,\sigma)$ vanishes at $z=1$ and also, for odd d, at $z=-1$.

It is also a fact that the lower half coefficients are all negative and the upper half all positive [7],

$$\begin{aligned} w_i &< 0\,, \quad i = 0,1,\ldots,d/2-1 \\ w_i &> 0\,, \quad i = d/2,1,\ldots,d-1\,, \end{aligned}$$

which constitutes our generalisation of the Weyl–Polya conjecture.

10. Conclusion

The main formal results are the Poincaré series (47), (48), for the coexact Laplacian degeneracies on d–dimensional fundamental domains. Some of the specific consequences, such as the termination of the heat–kernel expansion, are not

[7] Unfortunately I have not yet been able to prove this, but symbolic manipulation verifies it.

unexpected. Results involving S^3 generally extend, in some way, to higher odd dimensions.

I note that the heat–kernel expansions are all of the conventional form. Because of the fixed points, logarithmic terms might have been expected but it seems that these are hard to generate, [42], [43]. The ζ–function has the standard meromorphic structure and the implication is that the image method (the group average) automatically yields the Friedrichs extension.

The consequences of an irrational eta invariant for the signature have yet to be determined.

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